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Universal ratio of magnetization moments in two-dimensional Ising models

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Abstract. We calculate the universal critical-point ratios of the square of the second and the fourth moment of the magnetization for ferromagnetic Ising models on the square and on the triangular lattices. Periodic boundary conditions are used in accordance with the four-fold and six-fold rotational symmetries of the respective lattices. These results, which are obtained by means of an analysis of finite-size data computed with a transfer-matrix technique, have an accuracy of the order of one millionth. This analysis is also applied to rectangular systems with arbitrary aspect ratios.

1. Introduction

The critical exponents and critical couplings of a number of solvable two-dimensional Ising models are known exactly. There appears to be a wide class of models with the same critical exponents. This class can, by means of the universality hypothesis [1], be generalized to all two-dimensional Ising models with predominantly ferromagnetic, short-range pair interactions. However, it has been realized that, in addition to critical exponents, also certain critical-point amplitudes and ratios thereof are universal [2–4]. Finite-size scaling [5, 6] predicts that, for a system with finite size L and magnetization $M = \sum_i S_i$, the quantity

$$Q \equiv \lim_{L\to\infty} Q_L \equiv \lim_{L\to\infty} \langle M^2\rangle_L^2 / \langle M^4\rangle_L \tag{1}$$

is universal at the critical point [2], although dependent on the boundary conditions.

The ratio Q is a measure of the shape of the magnetization distribution; for instance, $Q = 1/3$ for a Gaussian distribution, and $Q = 1$ for the long-range ordered state. Its universal value for square, critical Ising systems with toroidal boundary conditions was determined by Bruce [7], and by Burkhardt and Derrida [8] as $Q \approx 0.86$. It is a very useful quantity for the determination of the critical points of models that are not solvable, but can be assumed to belong to the Ising universality class. The critical point of such a model can be estimated by expanding equation (1) about the critical point and by application of fitting procedures to Monte Carlo results for Q_L near criticality. First, we describe how this expansion is found. Since

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the moments of the magnetization distribution can be expressed in derivatives of the free energy with respect to the magnetic field, it is useful to start from the scaling relation for the free energy. We consider a system with finite-size parameter L, e.g. a hypercube of size L^d, or an elongated $L^{d-1} \times \alpha L$ system, with toroidal boundary conditions. In the vicinity of a renormalization fixed point, the parameters describing the system include the temperature field t, the magnetic field h and the finite-size field $1/L$. Thus, neglecting the irrelevant fields and nonlinearities, the asymptotic finite-size scaling relation for the singular part of the free energy per spin can be written [3, 6] as

$$f^{(s)}(t,h,1/L) = b^{-d} f^{(s)}(tb^{y_T}, hb^{y_H}, b/L) \tag{2}$$

where b is the rescaling factor, and y_T and y_H are the bulk thermal and magnetic critical exponents, respectively. This finite-size scaling relation has been widely applied in numerical studies of critical phenomena and in analysing experimental data, see e.g. [6]. Choosing $b = L$, differentiating k times to h and putting $h = 0$ in equation (2) yields

$$f^{(s),k}(t,0,1/L) = L^{ky_H - d} f^{(s),k}(tL^{y_T}, 0, 1). \tag{3}$$

The scaling function on the right hand side represents a system far from criticality (the finite-size parameter has the value 1) and is therefore assumed analytic, and may be Taylor expanded in tL^{y_T}. Expressing the magnetization moments $\langle M^k \rangle$ in derivatives of the free energy, neglecting the analytic part, one shows that these moments obey the same scaling behaviour as equation (3). Thus, in the vicinity of the fixed point (L large, t small) the ratio $Q_L(t)$ satisfies

$$Q_L(t) = Q + a_1 t L^{y_T} + a_2 t^2 L^{2y_T} + \cdots \tag{4}$$

where Q and the a_i are (in principle) unknown parameters. Another unknown is the critical temperature T_c, which enters into this expression by $t \sim T - T_c$. The unknowns can be determined by fitting equation (4) to the Monte Carlo data. If Q is known, one unknown parameter is eliminated, so that the critical point can be obtained more accurately.

The ratio Q depends not only on the type of boundary conditions [8] but also on the aspect ratio and, in the case of anisotropic couplings, on the ratio of the coupling strengths in different directions. Thus we consider a system with a reduced Hamiltonian

$$-\beta\mathcal{H} = \sum_{x=1}^{L_x} \sum_{y=1}^{L_y} (K_x s_{x,y} s_{x+1,y} + K_y s_{x,y} s_{x,y+1} + h s_{x,y}) \tag{5}$$

with toroidal boundaries: $s_{L_x+1,y} \equiv s_{1,y}$ and $s_{x,L_y+1} \equiv s_{x,1}$. In particular we focus on the critical point $h = 0$, $\sinh 2K_x \sinh 2K_y = 1$. In the case of isotropic couplings ($K = K_x = K_y$), the fixed-point Hamiltonian depends only on the finite size $L \equiv L_y$ and the aspect ratio $\alpha \equiv L_x/L_y$. Thus Q is, in this case, a universal function $Q(\alpha)$ of α. For a system with anisotropic couplings, the fixed-point Hamiltonian density will exhibit a similar anisotropy, which is marginal under rescaling. Thus, Q is now given by a different universal function $Q'(\alpha)$ of α. However, the anisotropy in the

Hamiltonian may be suppressed by means of an anisotropic scale transformation. In the case that the main directions correspond with the periodic boundaries, this translates in a change to the aspect ratio by a constant factor, say β, so that $L_x \to L'_x = L_x/\beta$, $L_y \to L'_y = L_y$, and $\alpha \to \alpha' = \alpha/\beta$. By equating $Q'(\alpha)$, which may be determined e.g. by Monte Carlo simulations, to $Q(\alpha/\beta)$, the renormalized anisotropy factor β can be solved.

Conformal invariance has been used [8] to calculate Q, but unfortunately this approach is restricted to rather special boundary conditions. On the other hand, the transfer-matrix results of [8] allow a rather accurate determination of Q for square systems with toroidal boundary conditions: graphical extrapolation of the data in table II of [8] yields $Q = 0.856 \pm 0.002$. This value was used and quoted e.g. [9] for the determination of the critical line of the simple quadratic Ising model with crossing bonds, and in [10] for the location of the critical point of self-dual Ising models with multi-spin interactions and a magnetic field. It compares well with the Monte Carlo result $Q = 0.855 \pm 0.001$ [11]. For very accurate applications, it would be helpful to know Q to an accuracy of more than 3 decimal places.

The aim of this paper is an accurate calculation of Q for the two-dimensional Ising model in a square and in a triangular geometry, as well as in a rectangular geometry with an arbitrary aspect ratio, with periodic boundary conditions in all cases. The second and fourth moments of the magnetization M occurring in (1) are calculated by means of a transfer-matrix technique, in combination with a perturbation expansion [8, 12]. This avoids numerical differentiation of the partition function, so that highly accurate finite-size data can be obtained. To determine Q for non-rational aspect ratios α, systems (5) with isotropic couplings ($K_x = K_y$) and a rectangular ($L_x \times L_y$) geometry are mapped onto a $\alpha L_x \times L_y$ system with anisotropic couplings, thus also providing an independent check for our direct calculations with isotropic couplings. This procedure can also be used to improve the finite-size convergence. Finite-size data for Q_L are then extrapolated by recourse to the power-law dependence inferred from the finite-size scaling analysis of $\langle M^2 \rangle_L$ and $\langle M^4 \rangle_L$.

The outline of the paper is as follows. In section 2 we apply finite-size scaling in order to obtain the powers of the system size L, occurring in the expansion of Q_L. This knowledge is useful for accurate extrapolations to infinite L. In section 3 we map the rectangular system (5) with isotropic interactions onto a system with anisotropic interactions, in order to enable a calculation of Q for non-rational aspect ratios. Section 4 deals with technical aspects of the transfer-matrix calculations and presents their results. Conclusions concerning the internal consistency, universality, applicability and the large-α limiting behaviour of the results are given in section 5.

2. Asymptotic finite-size dependence

The finite-size scaling relation equation (2) has to be modified for the two-dimensional Ising model, which is particular in some respects. Firstly, the specific heat singularity has a logarithmic divergence so that (see e.g. [13]) the zero-field free energy should contain explicitly a logarithmic term. Secondly, it has been argued [14] that the leading corrections to scaling are analytic and can be accounted for by nonlinearities of the scaling fields g_t and g_h related to the thermal field t and the ordering field h. It can be arranged that the renormalization equations, which are nonlinear in t and

h, become linear in variables g_t and g_h [15]. Thus, under rescaling by a factor of b

$$g_t' = b^{y_T} g_t \qquad g_h' = b^{y_H} g_h \tag{6}$$

where the primes denote renormalized quantities. Furthermore, we have for the finite-size field

$$1/L \to 1/L' = b/L$$

so that for the two-dimensional Ising model (5), which has $y_T = 1$, the fields g_t and $1/L$ fulfil the same relation. Thus their ratio $g_t L$ is invariant under rescaling and, along a trajectory with constant $g_t L$ one may combine g_t and $1/L$ into a single field proportional to $1/L$, keeping in mind that the critical amplitudes may still depend on $g_t L$. Along this trajectory the known results [14] for the scaling behaviour of the free energy in terms of g_t and g_h can be generalized and the corresponding singular part $F^{(s)}$ of the total free energy is thus expressed as

$$F^{(s)}(g_t, g_h, L^{-1}) = A(g_t L) \ln L + B(g_t L, g_h L^{y_H}) \tag{7}$$

where A and B are unknown amplitudes. The nonlinear fields are expanded as

$$\begin{aligned} g_t &= t + b_t h^2 + c_t t^2 + \cdots \\ g_h &= h(1 + c_h t + d_h t^2 + e_h h^2 + \cdots) . \end{aligned} \tag{8}$$

The scaling form of the free energy enables the calculation of derivates with respect to the field at the critical point $t = 0$, $h = 0$. Including an analytic contribution, we obtain

$$\begin{aligned} \left(\frac{\partial^2 F}{\partial h^2}\right)_{h=0} &= B^{(0,2)}(g_t L, 0) L^{2y_H} + 2A^{(1)}(g_t L) b_t L \ln L \\ &\quad + 2B^{(1,0)}(g_t L, 0) b_t L + cL^2 + \cdots \end{aligned} \tag{9}$$

where the superscripts of A and B denote derivatives, and

$$\left(\frac{\partial^4 F}{\partial h^4}\right)_{h=0} = \alpha_1 L^{4y_H} + \alpha_2 L^{2y_H} + \alpha_3 L^2 + \alpha_4 L^2 \ln L + \cdots . \tag{10}$$

From the relations

$$\langle M^2 \rangle = \frac{\partial^2 F}{\partial h^2} \tag{11}$$

$$\langle M^4 \rangle = \left[\frac{\partial^4 F}{\partial h^4} + 3\left(\frac{\partial^2 F}{\partial h^2}\right)^2\right] \tag{12}$$

we obtain the following expansion for Q_L up to L^{3-4y_H}

$$\begin{aligned} Q_L = Q_\infty &+ \beta_0 L^{2-2y_H}(1 + \beta_1 L^{-1} \pm \beta_2 L^{2-2y_H} + \beta_3 L^{-2} + \beta_4 L^{1-2y_H} + \cdots) \\ &+ \gamma_0 L^{1-2y_H} \ln L (1 + \gamma_1 L^{2-2y_H} + \cdots) . \end{aligned} \tag{13}$$

The finite-size expansion (13) contains not only a number of algebraic powers, but also logarithmic terms which may complicate the determination of Q from the finite-size results.

3. Anisotropic systems

We consider a square Ising lattice with isotropic couplings $K_x = K_y = K$ (equation (5)) with L_y rows and L_x columns. The same scaling relation [3] as before is applicable, but the scaling function, and thereby the universal quantity given by equation (1), will now depend on the aspect ratio $\alpha = L_x/L_y$.

At distances x, y that are much larger than the lattice unit and much smaller than the system sizes, the spin–spin correlation function decays algebraically at criticality

$$g_x(x) = a_x x^{-\eta} \qquad g_y(y) = a_y y^{-\eta} \tag{14}$$

where $a_x = a_y$, and the critical exponent $\eta = 1/4$ for the present system. After rescaling in the horizontal direction, ($x' = x/\beta$, $y' = y$) the aspect ratio has changed into $\alpha' = \alpha/\beta$, and the correlations have become anisotropic

$$g'_x(x') = a'_x x'^{-\eta} \qquad g'_y(y') = a'_y y'^{-\eta}\,. \tag{15}$$

The requirement that these correlations should be proportional to those before rescaling, leads to

$$a'_x = a'_y \beta^{\eta}\,. \tag{16}$$

Our purpose is to find an explicit Hamiltonian that reproduces the asymptotic behaviour of equations (15) and (16). Such a Hamiltonian should renormalize to the same fixed point as the anisotropically rescaled system. For the system described by equation (5), the amplitudes a_x, a_y are exactly known [16] and it follows that

$$\sinh 2K_x = \beta = (\sinh 2K_y)^{-1}\,. \tag{17}$$

More generally, the anisotropic scale reduction by a factor β relates a critical system with couplings K_x, K_y and aspect ratio α to one described by the following primed parameters:

$$\begin{aligned} \sinh 2K'_x &= \beta \sinh 2K_x \\ \sinh 2K'_y &= \beta^{-1} \sinh 2K_y \\ \alpha' &= \beta^{-1}\alpha. \end{aligned} \tag{18}$$

This approach is not restricted to rational $\alpha' = \alpha/\beta$, so the aspect ratio α can be considered a continuous variable. It enables us also to perform consistency checks on the numerical results. For example, we have verified that the result for Q on a square lattice with $\alpha = 2/\sqrt{3}$ agrees with that on a triangular lattice with rectangular periodic boundary conditions and the same aspect ratio.

4. Numerical technique and results

Our system consists of $L_x = \alpha L$ columns containing $L_y = L$ spins on a square lattice with periodic boundaries. The spins belonging to the jth column are denoted by $S_j = (S_{j1}, S_{j2}, \cdots, S_{jL_y})$ so that

$$Z = \sum_{S_1,\cdots,S_{L_x}} \exp[-\beta\mathcal{H}(S_1,\cdots,S_{L_x})] = \mathrm{Tr}\,\mathbf{T}^{L_x} \tag{19}$$

where the transfer matrix T is defined in the 2^{L_y}-dimensional manifold of all configurations of L_y spins, $S_{jk} = \pm 1$, $k = 1, \cdots, L_y$. The matrix T can be split into a product

$$\mathsf{T} = \mathsf{T}_v \mathsf{T}_h \tag{20}$$

where

$$T_v(S_i, S_j) = \exp\left[\sum_{k=1}^{L_y} (K_y S_{jk} S_{j,k+1} + h S_{ik})\right] \delta_{S_i,S_j} \tag{21}$$

is diagonal. The non-diagonal matrix

$$T_h(S_i, S_j) = \exp(K_x \sum_{k=1}^{L_y} S_{ik} S_{jk}) \tag{22}$$

can be expressed as a product of sparse matrices [17, 18] which is convenient for the numerical calculations.

Expanding the partition function in powers of h

$$Z(h) = \sum_{k=0}^{\infty} \frac{h^k}{k!} Z_k \tag{23}$$

where

$$Z_k = \left(\frac{\partial^k Z}{\partial h^k}\right)_{h=0} \tag{24}$$

the second and fourth moments of the magnetization can be evaluated from

$$\langle M^k \rangle_{h=0} = Z_k / Z_0.$$

Thus, to calculate Q_L, it suffices to find the corresponding coefficients in the expansion (23).

The evaluation of the trace in (19) requires a number of successive multiplications of a given vector v by T. Knowing the expansion of the sparse matrices in powers of h, one can perform the multiplication while keeping track of the power of h of each term. As far as the matrix T is concerned, the field dependence is restricted to the diagonal matrix T_v, which can be expanded as

$$\mathsf{T}_v(h) = \sum_{k=0}^{\infty} \frac{h^k}{k!} \mathsf{T}_v^{(k)}(0) \tag{25}$$

where

$$\begin{aligned} \mathsf{T}_v^{(j)}(0) &= \mathsf{S}^j \mathsf{T}_v(0) \\ S(S_i, S_j) &= \left(\sum_{k=0}^{L_y} S_{ik}\right) \delta_{S_i,S_j}. \end{aligned} \tag{26}$$

Denoting the resulting vector, after j multiplications of v by T, as $v^{(j)}$

$$v^{(j)} = \mathsf{T}^j v$$

and expressing $v^{(j)}$ in powers of h

$$v^{(j)} = \sum_{k=0}^{\infty} \frac{h^k}{k!} v_k^{(j)} \tag{27}$$

one observes that the vector $v^{(j+1)}$, obtained by a new application of T, can be uniquely expressed in terms of the coefficients present in (25) and (27). This perturbative scheme [8, 12] yields more accurate results for Q_L than numerical differentiation.

The numerical results for square ($L \times L$) systems are given in table 1 for $L \leqslant 17$. These Q_L values coincide with those published by Burkhardt and Derrida [8] ($L \leqslant 14$) up to all the decimal places quoted.

Table 1. The finite-size ratios Q_L for square and triangular Ising models with corresponding periodic boundary conditions. Results are also included for the non-interacting hard square lattice gas in a square geometry; the boundary conditions impose a restriction to even L. The extrapolated values ($L = \infty$) are shown in each of these cases. Estimated numerical uncertainties are shown in parentheses.

L	Square	Triangular	Hard squares
2	0.893 425 005 846 5210	0.899 082 568 807 3393	0.794 052 980 225 6821
3	0.877 531 433 858 0798	0.878 403 443 937 7256	
4	0.870 775 182 402 8686	0.870 523 452 495 5309	0.854 055 512 944 1982
5	0.866 871 585 857 0995	0.866 648 897 570 8098	
6	0.864 355 571 518 2825	0.864 448 525 064 1061	0.861 065 884 923 2626
7	0.862 642 460 185 1471	0.863 073 825 070 4151	
8	0.861 427 465 555 8456	0.862 154 238 452 5454	0.861 699 052 729 3661
9	0.860 535 529 603 1246	0.861 506 787 904 9795	
10	0.859 861 199 230 4495	0.861 032 485 481 7047	0.861 248 560 628 0001
11	0.859 338 485 960 6425	0.860 673 857 363 4970	
12	0.858 924 610 312 7896	0.860 395 604 274 4663	0.860 642 912 241 4773
13	0.858 590 919 771 5020	0.860 175 032 954 7905	
14	0.858 317 652 600 5431	0.859 996 991 895 0579	0.860 081 755 599 3912
15	0.858 090 820 107 1752	0.859 851 037 685 3666	
16	0.857 900 290 399 7150	0.859 729 773 792 0098	0.859 601 889 076 4437
17	0.857 738 574 853 2215		
⋮			
∞	0.856 216(1)	0.858 725 28(3)	0.856 25(5)

Having thus obtained Q_L with an accuracy limited only by the use of double-precision floating-point arithmetic, we have extrapolated the data using the asymptotic formula (13) for $L \to \infty$. We have performed direct fits of this expression, taking into account different numbers of terms. Another procedure is to construct a new series, which eliminates both the successive powers and the logarithmic term from (13), and then performing again power-law extrapolations. A given power L^{-x} is eliminated in the new series defined as

$$Q'_L = [(L+1)^x Q_{L+1} - L^x Q_L]/[(L+1)^x - L^x]$$

which still satisfies equation (13) with the same constant Q_∞, but without the term containing L^{-x}.

We have also applied the alternating ε-algorithm described in [6], where no explicit L-dependence is exploited. The results of this algorithm are consistent with the power-law fits. However, the convergence is less good, and the uncertainties are estimated to be one order of magnitude larger than those in the case of the power-law extrapolations.

For the triangular Ising model we employ a transfer matrix $\mathbf{T}$ that is very similar to one used for the square lattice in the diagonal direction [19]. The only difference is that an extra diagonal matrix is inserted (see equation (21)), thus accounting for the vertical bonds. Following the same perturbative approach, we have obtained finite-size results for Q_L ($L \leqslant 16$), which are also included in table 1. We emphasize that, in this case, the toroidal boundary conditions are not imposed in perpendicular directions, but in directions following the triangular symmetry of the lattice. Each site has 6 periodic images at a distance L, separated by angles $\varphi = \pi/3$. This is in contrast with the square system, which has 4 periodic images at a distance L separated by angles $\varphi = \pi/2$. Because of their different shapes, one may expect different values of Q in both systems. Indeed, our results confirm that this is the case.

In the last column of table 1 we present finite-size results for the Q-value of the non-interacting hard square lattice gas [20] on square systems with toroidal boundaries. This model is assumed to belong to the Ising universality class. Its critical point is not exactly known, but accurate estimates are available [19–21]. The numerical technique is described in [19]. In this case, the quantity M stands for the staggered density $\sum_{x,y} n_{x,y}(-1)^{x+y}$ where $n_{x,y} = 0$ and 1 describe an empty and an occupied site respectively. Only even L values are practical in view of the symmetry between the sublattices. Despite the small number of entries and their non-monotonic behaviour, the extrapolated value Q_∞ agrees quite well with that of the square Ising model.

Finally, results for Q of some rectangular systems with different aspect ratios α are shown in table 2. For $\alpha \leqslant 2$ the calculations used $L \times 3L$ systems with $L \leqslant 14$ ($L \leqslant 15$ for $\alpha = 1$) and couplings according to equation (18) and $\beta = \alpha/3$. This procedure leads to a better finite-size convergence than that using $L \times L$ systems. As a consequence, the $\alpha = 1$ result in table 2 is somewhat more accurate than that in table 1. For $\alpha > 2$ the calculations used $\alpha L \times L$ systems with L at least up to 11.

We have estimated, using power-law fits in $1/\alpha$ to the data for large aspect ratios, that

$$\lim_{\alpha\to\infty} Q(\alpha) = 0.333\,333(3)$$

which agrees with the value 1/3 for the Gaussian distribution describing linear systems.

Defining, for the strip geometry, the quantity [2, 8]

$$U_{L_y} = \lim_{L_x\to\infty} [L_x(1 - \tfrac{1}{3}\langle M^4\rangle_{L_x,L_y} \langle M^2\rangle^{-2}_{L_x,L_y})] \tag{28}$$

Table 2. Numerical results for the universal functions $Q(\alpha)$ and $\alpha U(\alpha)$ as well as for the coefficients a_i in the power series defined in equation (35). Estimated numerical uncertainties in the last decimal place are quoted in parentheses. A result for A_U obtained by means of a conformal mapping [8], is shown for comparison as the last entry in the sixth column.

α	$Q(\alpha)$	α	$Q(\alpha)$	α	$\alpha U(\alpha)$	i	a_i
1.00	0.856 2157(5)	10	0.441 345(1)	10	2.447 330(2)	0	+0.669 650 61
1.25	0.851 947(1)	15	0.398 7253(3)	15	2.460 04(1)	1	+0.323 776 92
1.50	0.841 5515(7)	20	0.380 0929(2)	20	2.460 43(1)	2	−0.157 016 79
1.75	0.827 049(1)	30	0.363 112(8)	30	2.460 3(6)	3	−0.018 784 47
2.00	0.809 678(3)	40	0.355 179(6)	40	2.460 2(7)	4	+0.090 941 36
3.00	0.728 090(3)	50	0.350 584(5)	50	2.460 3(9)	5	−0.119 223 24
4.00	0.650 069(3)	60	0.347 586(4)	60	2.460 3(6)	6	+0.066 181 00
5.00	0.587 126(2)	70	0.345 476(3)	70	2.460 3(6)	7	+0.094 318 46
6.00	0.539 396(2)	80	0.343 910(3)	80	2.460 3(7)	8	−0.122 306 77
7.00	0.503 811(1)	90	0.342 702(2)	90	2.460 4(5)	9	−0.008 471 09
8.00	0.477 176(1)	100	0.341 742(1)	100	2.460 5(3)	10	+0.047 335 66
9.00	0.456 964(1)	∞	0.333 333(3)	∞	2.460 44(2)	11	−0.010 180 46

the ratio

$$A_U = \lim_{L\to\infty} [L^{-1} U_L] \tag{29}$$

is also universal. This amplitude reflects that the magnetization distribution of a strip with a finite length deviates from the Gaussian distribution. An interchange of the limits in equations (29) and (28) leads to

$$A_U = \lim_{\alpha\to\infty} \alpha U(\alpha) \tag{30}$$

where

$$U(\alpha) = \lim_{L_x,\, L_y\to\infty,\, L_x/L_y=\alpha} \left(1 - \tfrac{1}{3}\langle M^4\rangle_{L_x,L_y} \langle M^2\rangle^{-2}_{L_x,L_y}\right). \tag{31}$$

In table 2 we report our results for $A_U(\alpha) = \alpha U(\alpha)$, and show their consistency with the value A_U determined from conformal invariance [8]. The latter value refers to a different set of boundary conditions than those in the present case; therefore we comment on the possibility that A_U depends on the boundary conditions in the length direction of the system.

Denoting the boundary weight by B_{S_1,S_L}, the partition function of the model with arbitrary boundaries is

$$Z = \sum_{S_1,S_L=1}^{2^{L_y}} [\mathbf{T}^{L-1}]_{S_1,S_L} B_{S_1,S_L} = \sum_{S_1,S_L=1}^{2^{L_y}} [\mathbf{R}\Lambda^{L-1}\mathbf{L}]_{S_1,S_L} B_{S_1,S_L} \tag{32}$$

where $\Lambda(i,j) = \lambda_i \delta_{i,j}$ is a diagonal matrix containing the eigenvalues of $\mathbf{T}$, and $\mathbf{R}$ and $\mathbf{L}$ are matrices built up of the corresponding right and left eigenvectors of

T, respectively, and normalized such that **LR** = 1. Ordering the eigenvalues in decreasing absolute value, an elongated system ($L_x >> L_y$) will obey

$$Z_{L_x,L_y} = \gamma_1(h)[\lambda_1(h)]^{L_x} + \cdots \tag{33}$$

where the coefficient $\gamma_1(h)$ depends only on the elements of **B**, **R** and **L**. The dots represent exponentially small corrections when $L_x \to \infty$. Thus, using $F = \ln Z$ and equations (11) and (12), the quantity U_L (equation (28)) can be expressed in the derivatives of $\gamma_1(h)$ and $\lambda_1(h)$. It is then easily verified that A_U is independent of the boundary conditions.

From table 2 we see that $A_U(\alpha)$ reaches the asymptotic region much faster than $Q(\alpha)$. $A_U(\alpha)$ reaches the value A_U for $\alpha \geqslant 20$, whereas $Q(\alpha)$ deviates significantly from the asymptotic value $\frac{1}{3}$ even for $\alpha = 100$. In the absence of boundary terms (i.e., periodic boundaries), $A_U(\alpha)$ approaches A_U exponentially fast, so that for α large enough we can put

$$Q(\alpha) = \tfrac{1}{3}(1 - A_U/\alpha)^{-1}. \tag{34}$$

Indeed, $Q(\alpha)$ as calculated from formula (34), agrees with the values quoted in table 2 within $\pm 1 \times 10^{-5}$ for $15 \leqslant \alpha \leqslant 20$, and within the estimated numerical uncertainty for $\alpha \geqslant 20$. Thus equation (34) adequately describes $Q(\alpha)$ for $\alpha \geqslant 15$.

Our results for $Q(\alpha)$ are, within a margin of 3×10^{-5}, reproduced by the expression

$$Q(\alpha) \approx \sum_{i=0}^{11} a_i A^i \tag{35}$$

where $A = 4/(\alpha + 1/\alpha) - 1$. The coefficients a_i were determined by least-squares fits and are given in the last column of table 2. Although we have no compelling reasons to choose the special form of equation (35), we note that it is analytic for positive α and satisfies the symmetry $\alpha \leftrightarrow \alpha^{-1}$, and that the expansion parameter A satisfies $|A| \leqslant 1$.

The ratio Q is plotted in figure 1 on a semi-logarithmic scale versus the aspect ratio α. The smooth curve represents equation (35).

5. Conclusions

We have numerically calculated the universal ratio Q from the second and fourth moments of the magnetization distribution of the Ising model on square and triangular lattices with toroidal boundary conditions following the lattice symmetry. The value of Q is somewhat different for the two systems; this difference is due to the different boundary conditions and has nothing to do with the lattice structure. The ratio Q was also calculated for toroidal systems with square symmetry of the non-interacting hard-square lattice gas. The result is in agreement with universality.

For boundary conditions with rectangular symmetry, we present a function approximating $Q(\alpha)$ for all aspect ratios α. The numerical results, which were obtained for values of α up to 100, approach the limit 1/3 when $\alpha \to \infty$, in a

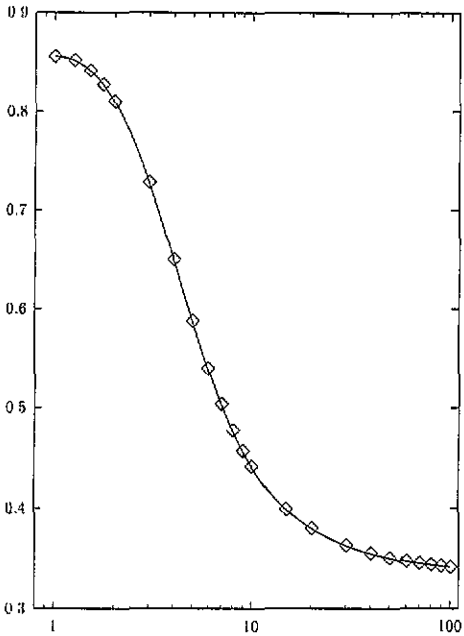

Figure 1. The universal function $Q(\alpha)$ as a function of the aspect ratios α in the extended interval $1 \leqslant \alpha \leqslant 100$, plotted on a semi-logarithmic scale. The curve represents equation (35) and interpolates through the discrete data shown by the symbol $\diamond$.

way that correctly reproduces the quantity A_U derived in [8]. It is verified that the universal amplitude A_U for elongated systems does not depend on the boundary conditions in the length direction.

The high precision of the extrapolated results for Q reflects the accuracy of finite-size data obtained by the perturbation expansion method, and the utilization of finite-size scaling results that restrict the form of the large L-asymptotic behaviour of Q_L.

These results for Q are applicable in the Monte Carlo determination of critical points and, in the case of anisotropic models, the determination of renormalized aspect ratios of Ising-like models.

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